

1.5 Greatest Common Factor and Least Common Multiple

This chapter will conclude with two topics which will be used when working with fractions. Recall that factors of a number are numbers that divide into it. For example, the factors of the numbers 18 and 24 are:

factors of 18: 1, 2, 3, **6**, 9, 18
factors of 24: 1, 2, 3, 4, **6**, 8, 12, 24

The **greatest common factor** (abbreviated GCF) is the largest common factor. That is, it is the largest factor common to both lists. A quick survey of the factors indicates the GCF of 18 and 24 must be 6, since it is a factor in each list, and it is the largest such factor common to both lists. If three numbers are used, we still find the common factor to all three lists. For example, the factors of the numbers 24, 32, and 80 are:

factors of 24: 1, 2, 3, 4, 6, **8**, 12, 24
factors of 32: 1, 2, 4, **8**, 16, 32
factors of 80: 1, 2, 4, 5, **8**, 10, 16, 20, 40, 80

The GCF of 24, 32, and 80 must be 8, since it is the largest number common to both lists.

Example 1 Find the greatest common factor of each set of numbers by listing factors.

- a. 12, 18
- b. 24, 30
- c. 40, 60
- d. 12, 24, 40

Solution a. Begin by listing the factors of 12 and 18:

factors of 12: 1, 2, 3, 4, **6**, 12
factors of 18: 1, 2, 3, **6**, 9, 18

Since 6 is the largest number common to both lists, the GCF of 12 and 18 is 6. Note that 2 and 3 are also common factors, but they are not the *greatest* common factor (largest such common factor).

- b. Begin by listing the factors of 24 and 30:

factors of 24: 1, 2, 3, 4, **6**, 8, 12, 24
factors of 30: 1, 2, 3, 5, **6**, 10, 15, 30

Since 6 is the largest number common to both lists, the GCF of 24 and 30 is 6.

- c. Begin by listing the factors of 40 and 60:

factors of 40: 1, 2, 4, 5, 8, 10, **20**, 40
factors of 60: 1, 2, 3, 4, 6, 10, 15, **20**, 30, 60

Since 20 is the largest number common to both lists, the GCF of 40 and 60 is 20.

- d. Begin by listing the factors of 12, 24, and 40:

factors of 12: 1, 2, 3, **4**, 6, 12
factors of 24: 1, 2, 3, **4**, 6, 8, 12, 24
factors of 40: 1, 2, **4**, 5, 8, 10, 20, 40

Since 4 is the largest number common to all three lists, the GCF of 12, 24, and 40 is 4.

Though this method of finding the GCF seems reasonable, when the numbers get larger it can become quite difficult to list all of the factors of two numbers. For example, suppose we wanted to find the GCF of 84 and 120. We will investigate an approach using prime numbers. Let's begin by finding the prime factorizations of 84 and 120:

$$84 = 4 \cdot 21 = (2 \cdot 2) \cdot (3 \cdot 7) = \mathbf{2 \cdot 2 \cdot 3 \cdot 7}$$

$$120 = 10 \cdot 12 = (2 \cdot 5) \cdot (4 \cdot 3) = (2 \cdot 5) \cdot (2 \cdot 2 \cdot 3) = \mathbf{2 \cdot 2 \cdot 2 \cdot 3 \cdot 5}$$

Recall that the GCF is a factor of each number. But remember from the previous section that all factors of a number must be composed of its primes. Thus the factors of 84 are all comprised of some combination of 2 (can be used twice), 3, and 7. Similarly, the factors of 120 are all comprised of some combination of 2 (can be used three times), 3, and 5. Since the GCF is common to both lists, how large can it be? It could have 2 (used twice) and 3, or $2 \cdot 2 \cdot 3 = 12$. Thus the GCF of 84 and 120 is 12. The procedure is the same using three numbers. Suppose we want to find the GCF of 48, 60, and 150. First find the prime factorizations of the three numbers:

$$48 = 4 \cdot 12 = (2 \cdot 2) \cdot (3 \cdot 4) = (2 \cdot 2) \cdot (3 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$60 = 10 \cdot 6 = (2 \cdot 5) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 5$$

$$150 = 10 \cdot 15 = (2 \cdot 5) \cdot (3 \cdot 5) = 2 \cdot 3 \cdot 5 \cdot 5$$

Looking at the three lists, we are looking for the largest set of primes common to all three lists. This is $2 \cdot 3 = 6$, so the GCF of 48, 60, and 150 is 6.

Example 2 Find the greatest common factor of each set of numbers by using primes.

- a. 48, 80
- b. 60, 150
- c. 168, 210
- d. 45, 60, 180

Solution a. Begin by finding the prime factorization of each number:

$$48 = 8 \cdot 6 = (2 \cdot 4) \cdot (2 \cdot 3) = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$80 = 10 \cdot 8 = (2 \cdot 5) \cdot (2 \cdot 4) = (2 \cdot 5) \cdot (2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$$

The largest set of primes common to both lists is $2 \cdot 2 \cdot 2 \cdot 2 = 16$, so the GCF of 48 and 80 is 16.

b. Begin by finding the prime factorization of each number:

$$60 = 10 \cdot 6 = (2 \cdot 5) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 5$$

$$150 = 10 \cdot 15 = (2 \cdot 5) \cdot (3 \cdot 5) = 2 \cdot 3 \cdot 5 \cdot 5$$

The largest set of primes common to both lists is $2 \cdot 3 \cdot 5 = 30$, so the GCF of 60 and 150 is 30.

- c. Begin by finding the prime factorization of each number:

$$168 = 4 \cdot 42 = (2 \cdot 2) \cdot (6 \cdot 7) = (2 \cdot 2) \cdot (2 \cdot 3 \cdot 7) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$$

$$210 = 10 \cdot 21 = (2 \cdot 5) \cdot (3 \cdot 7) = 2 \cdot 3 \cdot 5 \cdot 7$$

The largest set of primes common to both lists is $2 \cdot 3 \cdot 7 = 42$, so the GCF of 168 and 210 is 42.

- d. Begin by finding the prime factorization of each number:

$$45 = 9 \cdot 5 = (3 \cdot 3) \cdot 5 = 3 \cdot 3 \cdot 5$$

$$60 = 10 \cdot 6 = (2 \cdot 5) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 5$$

$$180 = 10 \cdot 18 = (2 \cdot 5) \cdot (3 \cdot 6) = (2 \cdot 5) \cdot (3 \cdot 3 \cdot 2) = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$$

The largest set of primes common to all three lists is $3 \cdot 5 = 15$, so the GCF of 45, 60, and 180 is 15.

Two numbers whose greatest common factor is 1 (that is, they have no prime factors in common) are called **relatively prime**. For example, the numbers 9 and 14 are relatively prime, since the prime factor of 9 is 3 and the prime factors of 14 are 2 and 7, thus there are no prime factors in common to the two lists.

The second topic in this section deals with **multiples** of numbers. Multiples of a number are values obtained from multiplication of that number by numbers. For example, the first six multiples of 12 are:

multiples of 12: 12, 24, 36, 48, 60, 72, ...

Now we define the **least common multiple** (abbreviated LCM) as the smallest common multiple of two different numbers. Given the two numbers 18 and 24, first list the multiples of each:

multiples of 18: 18, 36, 54, **72**, 90, 108, 126, 144, ...

multiples of 24: 24, 48, **72**, 96, 120, 144, ...

So 72 is the LCM of 18 and 24, since it is the smallest number common to both lists of multiples. Also note that 144 is common to both lists, however it is not the smallest number which is common (that is, it is not the *least* common multiple).

If three numbers are given the procedure is the same. Given the numbers 10, 15, and 25, first list the multiples of each number:

multiples of 10: 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, **150**, ...
multiples of 15: 15, 30, 45, 60, 75, 90, 105, 120, 135, **150**, 165, 180, 195, 210, ...
multiples of 25: 25, 50, 75, 100, 125, **150**, 175, 200, ...

Since 150 is the smallest number common to all three lists, the LCM of 10, 15, and 25 is 150.

Example 3 Find the least common multiple of each set of number by listing multiples.

- a. 12, 16
- b. 10, 25
- c. 45, 60
- d. 4, 6, 8

Solution a. Begin by listing the multiples of 12 and 16:

multiples of 12: 12, 24, 36, **48**, 60, 72, 84, ...
multiples of 16: 16, 32, **48**, 64, 80, 96, ...

Since 48 is the smallest number common to both lists, the LCM of 12 and 16 is 48.

b. Begin by listing the multiples of 10 and 25:

multiples of 10: 10, 20, 30, 40, **50**, 60, 70, 80, ...
multiples of 25: 25, **50**, 75, 100, 125, 150, ...

Since 50 is the smallest number common to both lists, the LCM of 10 and 25 is 50.

c. Begin by listing the multiples of 45 and 60:

multiples of 45: 45, 90, 135, **180**, 225, 270, ...
multiples of 60: 60, 120, **180**, 240, 300, 360, ...

Since 180 is the smallest number common to both lists, the LCM of 45 and 60 is 180.

d. Begin by listing the multiples of 4, 6, and 8:

multiples of 4: 4, 8, 12, 16, 20, **24**, 28, 32, ...

multiples of 6: 6, 12, 18, **24**, 30, 36, 42, ...

multiples of 8: 8, 16, **24**, 32, 40, 48, 54, ...

Since 24 is the smallest number common to all three lists, the LCM of 4, 6, and 8 is 24.

We will now develop an approach to find the least common multiple of two numbers using prime factorizations. Recall earlier we found the GCF of 84 and 120. The prime factorizations of those two numbers were:

$$84 = 4 \cdot 21 = (2 \cdot 2) \cdot (3 \cdot 7) = 2 \cdot 2 \cdot 3 \cdot 7$$

$$120 = 10 \cdot 12 = (2 \cdot 5) \cdot (4 \cdot 3) = (2 \cdot 5) \cdot (2 \cdot 2 \cdot 3) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$$

Any multiple of 84 must contain the primes of 84, thus any multiple of 84 must contain the primes $2 \cdot 2 \cdot 3 \cdot 7$. Similarly, any multiple of 120 must contain the primes of 120, thus it must contain the primes $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$. Since the LCM must be common to both list, it must contain both sets of primes. What is the smallest number possible? It must have three 2's (from the second list), a 3, a 5, and a 7, so it is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 840$. Thus the LCM of 84 and 120 is 840. The procedure is the same using three numbers. Suppose we want to find the LCM of 48, 60, and 150. Using the prime factorizations of the three numbers:

$$48 = 4 \cdot 12 = (2 \cdot 2) \cdot (3 \cdot 4) = (2 \cdot 2) \cdot (3 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$60 = 10 \cdot 6 = (2 \cdot 5) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 5$$

$$150 = 10 \cdot 15 = (2 \cdot 5) \cdot (3 \cdot 5) = 2 \cdot 3 \cdot 5 \cdot 5$$

For the LCM to be a multiple of 48, its prime factorization must contain four 2's and one 3. To be a multiple of 60, it must also contain a 5 (the four 2's and one 3 take care of the rest of 60). To be a multiple of 150, it needs another 5, so it is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = 1,200$. Thus the LCM of 48, 60, and 150 is 1,200.

Example 4 Find the least common multiple of each set of numbers by using primes.

- a. 56, 104
- b. 168, 210
- c. 126, 180
- d. 36, 90, 216

Solution a. Begin by finding the prime factorization of each number:

$$56 = 8 \cdot 7 = (2 \cdot 4) \cdot 7 = (2 \cdot 2 \cdot 2) \cdot 7 = 2 \cdot 2 \cdot 2 \cdot 7$$

$$104 = 4 \cdot 26 = (2 \cdot 2) \cdot (2 \cdot 13) = 2 \cdot 2 \cdot 2 \cdot 13$$

The smallest set of primes containing both lists is $2 \cdot 2 \cdot 2 \cdot 7 \cdot 13 = 728$, so the LCM of 56 and 104 is 728.

b. Begin by finding the prime factorization of each number:

$$168 = 4 \cdot 42 = (2 \cdot 2) \cdot (6 \cdot 7) = (2 \cdot 2) \cdot (2 \cdot 3 \cdot 7) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$$

$$210 = 10 \cdot 21 = (2 \cdot 5) \cdot (3 \cdot 7) = 2 \cdot 3 \cdot 5 \cdot 7$$

The smallest set of primes containing both lists is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 840$, so the LCM of 168 and 210 is 840.

c. Begin by finding the prime factorization of each number:

$$126 = 3 \cdot 42 = 3 \cdot (6 \cdot 7) = 3 \cdot (2 \cdot 3 \cdot 7) = 2 \cdot 3 \cdot 3 \cdot 7$$

$$180 = 10 \cdot 18 = (2 \cdot 5) \cdot (3 \cdot 6) = (2 \cdot 5) \cdot (3 \cdot 2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$$

The smallest set of primes containing both lists is $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 1,260$, so the LCM of 126 and 180 is 1,260.

d. Begin by finding the prime factorization of each number:

$$36 = 6 \cdot 6 = (2 \cdot 3) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3$$

$$90 = 10 \cdot 9 = (2 \cdot 5) \cdot (3 \cdot 3) = 2 \cdot 3 \cdot 3 \cdot 5$$

$$216 = 8 \cdot 27 = (2 \cdot 4) \cdot (3 \cdot 9) = (2 \cdot 2 \cdot 2) \cdot (3 \cdot 3 \cdot 3) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

The smallest set of primes containing both lists is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 = 1,080$, so the LCM of 36, 90, and 216 is 1,080.

The use of prime numbers throughout this section is not just as a convenience for working with larger numbers. It also serves as a bridge to algebra, where the primes will be replaced with variables x and y . For example, to find the GCF of $12x^2y^3$ and $8x^5y$, we are looking for the largest expression which is common to both lists. It must contain a 4, two x 's, and a y , so the GCF is $4x^2y$. Similarly, to find the LCM, we are looking for the smallest expression which contains both lists. It must contain 24, five x 's, and three y 's, so the LCM is $24x^5y^3$.

Example 5 Find the GCF and LCM of each expression.

- a. $7x^4y^5, 14x^3y^6$
- b. $10a^8b^5, 6a^{10}b^3$
- c. $4x^3y^2, 12x^5y$
- d. $6x^3y^4, 8x^5y^2, 10x^4y^3$

- Solution**
- a. The common factors are 7, x (three times), and y (five times), so the GCF is $7x^3y^5$. The smallest multiple must contain 14, x (four times), and y (six times), so the LCM is $14x^4y^6$.
 - b. The common factors are 2, a (eight times), and b (three times), so the GCF is $2a^8b^3$. The smallest multiple must contain 30 (the LCM of 10 and 6), a (ten times), and b (five times), so the LCM is $30a^{10}b^5$.
 - c. The common factors are 4, x (three times), and y , so the GCF is $4x^3y$. The smallest multiple must contain 12, x (five times), and y (two times), so the LCM is $12x^5y^2$.
 - d. The common factors are 2 (the GCF of 6, 8, 10), x (three times), and y (two times), so the GCF is $2x^3y^2$. The smallest multiple must contain 120 (which is the LCM of 6, 8, and 10), x (five times), and y (four times), so the LCM is $120x^5y^4$.

Terminology

greatest common factor (GCF)
multiple

relatively prime
least common multiple (LCM)

Exercise Set 1.5

Find the (a) greatest common factor and (b) least common multiple for each set of numbers by listing factors and multiples.

- | | |
|----------------|----------------|
| 1. 45, 60 | 2. 16, 20 |
| 3. 80, 100 | 4. 60, 90 |
| 5. 16, 30 | 6. 24, 30 |
| 7. 36, 54 | 8. 14, 63 |
| 9. 8, 12, 15 | 10. 10, 25, 40 |
| 11. 20, 30, 40 | 12. 18, 27, 36 |

Find the (a) greatest common factor and (b) least common multiple for each set of numbers by using primes.

- | | |
|-----------------|----------------|
| 13. 48, 72 | 14. 70, 170 |
| 15. 36, 40 | 16. 45, 81 |
| 17. 216, 234 | 18. 220, 242 |
| 19. 270, 320 | 20. 275, 286 |
| 21. 384, 432 | 22. 364, 468 |
| 23. 261, 290 | 24. 265, 424 |
| 25. 16, 18, 54 | 26. 15, 24, 40 |
| 27. 22, 65, 121 | 28. 16, 38, 76 |

Find the (a) greatest common factor and (b) least common multiple for each set of expressions.

- | | |
|-------------------------------|-----------------------------------|
| 29. $45x^2y^5, 15x^3y^3$ | 30. $30x^3y^7, 15x^6y^4$ |
| 31. $8a^7b^6, 6a^5b^2$ | 32. $12a^6b^5, 18a^4b^3$ |
| 33. $16x^8y^{12}, 20x^9y^7$ | 34. $36x^6y^{11}, 24x^9y^8$ |
| 35. $4x^2y^3, 6x^3y^4, 8x^4y$ | 36. $10x^5y^4, 5x^2y^2, 15x^3y^7$ |

Recall that two numbers are **relatively prime** if their GCF is 1. Determine whether each pair of numbers is relatively prime.

- | | |
|--------------|--------------|
| 37. 70, 75 | 38. 32, 84 |
| 39. 35, 72 | 40. 49, 93 |
| 41. 81, 192 | 42. 245, 246 |
| 43. 155, 248 | 44. 153, 272 |

An interesting result from number theory is the following theorem:

If A and B are two whole numbers, the product of their LCM and GCF is equal to their product AB .

Verify this theorem for the given pair of numbers.

- | | |
|-----------------------|------------------------|
| 45. 6 and 8 | 46. 12 and 16 |
| 47. 18 and 27 | 48. 24 and 30 |
| 49. 36 and 60 | 50. 45 and 60 |
| 51. 72 and 144 | 52. 150 and 300 |

Answer the following questions (based on the theorem stated above).

- 53.** If two numbers are relatively prime, what does this theorem tell us about their LCM?
- 54.** If the product of two numbers is equal to their LCM, what does this theorem tell us about the two numbers?
- 55.** Suppose the product of two numbers is 150 and their GCF is 5. What is the LCM of the two numbers?
- 56.** Suppose the product of two numbers is 144 and their GCF is 4. What is the LCM of the two numbers?
- 57.** Suppose the product of two numbers is 200 and their LCM is 40. What is the GCF of the two numbers?
- 58.** Suppose the product of two numbers is 160 and their LCM is 20. What is the GCF of the two numbers?
- 59.** True or False: The GCF of two numbers is a factor of their LCM.
Explain why/why not.
- 60.** True or False: The LCM of two numbers is a factor of their GCF.
Explain why/why not.

Solve the following application problems.

- 61.** John and Tracy are moving to a new house, and have a lot of books to move. John can carry 6 books at a time, while Tracy can carry 4. John wants to make 10 trips, then stop. How many trips will Tracy need to make to have carried the same number of books?
- 62.** Brad and Ross are sailing to Mexico. Ross can man the helm for 40 minutes before taking a break, and Brad can man the helm for 25 minutes before breaking. If Ross takes 10 breaks during their run, how many breaks does Brad take?
- 63.** Three satellites are vertically aligned above the North Pole. One satellite orbits the earth in 12 hours, another orbits the earth in 15 hours, while the third satellite orbits the earth in 8 hours. In how long will the satellites again be vertically aligned above the North Pole?

- 64.** Three policemen meet at Eddie's Donuts at 7:00 AM. One returns every 20 minutes, the second returns every 30 minutes, and the third returns every 25 minutes. At what time will the three policemen meet again at Eddie's Donuts?